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Brownian path integral from Dirac equation: a probabilistic approach to the Foldy–Wouthuysen transformation

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Abstract. The spectrum of the Dirac Hamiltonian H_{D} has a positive and a negative part, the first corresponds to the energy levels of the electron, the second to minus the energy levels of the positron. The Foldy–Wouthuysen transformation is a tool to obtain a decomposition of H_{D} in a direct sum $H_{\mathrm{D}} = H_{\mathrm{e}-} \oplus -H_{\mathrm{e}+}$, where $H_{\mathrm{e}-}$ is the Hamiltonian of the electron and $H_{\mathrm{e}+}$ is that of the positron. Unfortunately, this decomposition has been obtained in a closed form only when the external field is purely magnetic, while, in the presence of an electric field, only a perturbative expansion is available.

In this paper we give a path integral representation of the semigroup $\phi(\boldsymbol{x}, t) = \exp\{-(t/\hbar)H_{\mathrm{e}-}\}\phi_0(\boldsymbol{x})$ in an external electromagnetic field. Our formula is the relativistic version, for Dirac particles, of the well known Feynman–Kac–Itô formula for Schrödinger semigroups. The result can also be regarded as a tool to obtain the decomposition $H_{\mathrm{D}} = H_{\mathrm{e}-} \oplus -H_{\mathrm{e}+}$ even in the presence of a non-trivial electric field.

1. Introduction

The Feynman–Kac formula [1, 2] is a path integral representation of the semigroup $\psi(t, \boldsymbol{x}) = \exp\{-(t/\hbar)H_{\mathrm{S}}\}\psi_0(\boldsymbol{x})$ where H_{S} is a Schrödinger Hamiltonian. Due to its smoothing properties, the Hamiltonian semigroup is a far more useful object than the unitary group $\psi(t, \boldsymbol{x}) = \exp\{-\mathrm{i}(t/\hbar)H_{\mathrm{S}}\}\psi_0(\boldsymbol{x})$ which provides the (real) time evolution of the quantum system (see [3] for a discussion about this important point), hence the interest in working in the Euclidean region. The Feynman–Kac formula fully controls the Schrödinger semigroup and it is, therefore, a powerful tool with which to obtain information on the spectrum and on the eigenfunctions of H_{S} (see, for instance, [3–5]).

Finding an analogous path integral representation of the Dirac propagator for real or imaginary time has been an open problem for a long time. Historically, the first approaches were based on zigzag paths in spacetime along which the Dirac particle travels at the speed of light. This idea goes back to Feynman and Riazanov [6, 7] but, in spite of many improvements [8–17], it has not given a completely satisfactory answer to the problem. In some of these papers, in fact, additional assumptions are requested (for example analytic continuation of constants such as electron mass and light speed, or a lattice version of the spacetime). Furthermore, these approaches,

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which rely on the paths of ordinary stochastic processes in $1+1$ dimensions, need paths of non-commuting variables in the $(1+3)$-dimensional case (see [11]).

The papers [16, 17] in which a probabilistic representation of $\psi(t, \boldsymbol{x}) = \exp\{-\mathrm{i}(t/\hbar)H_D\}\psi_0(\boldsymbol{x})$ without analytical continuation of constants or discretizations is given merit special mention. The result holds for an electron in an external electromagnetic field in $1+1$ dimensions; furthermore, it has been extended to $1+3$ dimensions in special cases (for example central electric fields and spherical symmetric wavefunctions).

Our approach is radically different because we think that the correct semigroup (we work in the Euclidean region) to be considered should not directly involve the Dirac Hamiltonian, but only its positive part. It is well known that the spectrum of the Dirac Hamiltonian H_{D} has a positive and a negative part, the first corresponding to the energy levels of the electron, the second to minus the energy levels of the positron. By means of the Foldy–Wouthuysen transformation [18], one can obtain, in principle, a decomposition of H_{D} as a direct sum $H_{\mathrm{D}} = H_{\mathrm{e}^-} \oplus -H_{\mathrm{e}^+}$, where H_{e^-} is the Hamiltonian of the electron and H_{e^+} is that of the positron. This task can be performed in a closed form only when the external field is purely magnetic [19]. In this case one has the explicit operator H_{e^-} and can try to relate it to a generator of a stochastic process. This is exactly what we did in a previous paper [20] where we found a probabilistic representation involving diffusions and a Poisson process.

Unfortunately, when the electric field is not trivial, the Foldy–Wouthuysen transformation gives the Hamiltonian of the electron only as a perturbative expansion. This means that the operator H_{e^-} is not explicitly known in a closed form and the previous method, which worked in the purely magnetic case, can no longer be used. In this paper we bypass the problem and we give a path integral representation of the semigroup $\phi(t, \boldsymbol{x}) = \exp\{-(t/\hbar)H_{\mathrm{e}^-}\}\phi_0(\boldsymbol{x})$ which defines, albeit implicitly, the operator H_{e^-} in a non-perturbative way. Our path integral representation is still based on diffusions in the (Euclidean) spacetime and on a Poisson process which takes care of the spin of the particle. It holds for electrons in external electromagnetic fields which are only supposed to be not strong enough to eliminate the energy gap between the two halves of the Dirac spectrum (see [21, 22] for a discussion of this point). Our previous result for electrons in a purely magnetic field are found as a particular case.

Since the same method can be used to obtain the semigroup associated with the positron Hamiltonian, our technique can also be regarded as a general tool with which to obtain the decomposition $H_{\mathrm{D}} = H_{\mathrm{e}^-} \oplus -H_{\mathrm{e}^+}$ in a compact form. It turns out that our H_{e^-} is unitarily equivalent to the electron Hamiltonian which can be obtained from the Foldy–Wouthuysen transformation.

The paper is organized as follows: in section 2 we describe the path integral representation for the semigroup $\phi(t, \boldsymbol{x}) = \exp\{-(t/\hbar)H_{\mathrm{e}^-}\}\phi_0(\boldsymbol{x})$ without proof. In section 3 we discuss the single electron Hilbert space associated with the Hamiltonian H_{e^-} in our representation. In section 4 we compare our result with similar results for spinless relativistic particles and we discuss the non-relativistic limit. The proof is divided into two parts, contained in the appendices A and B.

2. The formula

Our starting point is the (Euclidean) Dirac equation in four spacetime dimensions

$$\hbar\frac{\partial\psi}{\partial t} = -H_{\mathrm{D}}\psi \tag{2.1}$$

where H_{D} is the usual Dirac Hamiltonian

$$H_{\mathrm{D}} = c\boldsymbol{\alpha}\cdot(-\mathrm{i}\hbar\nabla - e\boldsymbol{A}) + mc^2\beta + eA_0. \tag{2.2}$$

Here $\boldsymbol{A}$ and A_0 are the (time independent) electromagnetic potentials in the gauge $\nabla\cdot\boldsymbol{A} = 0$ and the four matrices $\boldsymbol{\alpha}$ and β are taken in the spinorial representation:

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

(the $\boldsymbol{\sigma}$s are the Pauli matrices). It is well known [23] that the first-order equation (2.1), for the four-component spinor ψ, is completely equivalent to a second-order equation for a Pauli spinor ϕ. In fact, by decomposing ψ as $\psi = \binom{\phi}{\chi}$, where ϕ and χ are two Pauli spinors, equation (2.1) splits into the pair

$$\begin{aligned} \hbar\frac{\partial\phi}{\partial t} &= c\boldsymbol{\sigma}\cdot(\mathrm{i}\hbar\nabla + e\boldsymbol{A})\phi - mc^2\chi - eA_0\phi \\ \hbar\frac{\partial\chi}{\partial t} &= -c\boldsymbol{\sigma}\cdot(\mathrm{i}\hbar\nabla + e\boldsymbol{A})\chi - mc^2\phi - eA_0\chi. \end{aligned} \tag{2.4}$$

The first of these two equations can be rewritten in the form

$$\chi = -\frac{1}{mc^2}\left[\hbar\frac{\partial}{\partial t} - c\boldsymbol{\sigma}\cdot(\mathrm{i}\hbar\nabla + e\boldsymbol{A}) + eA_0\right]\phi \tag{2.5}$$

and gives χ in terms of ϕ. Substituting χ in the second equation (and dividing by $2mc^2\hbar$) one has the (Euclidean) Feynman and Gell-Mann equation:

$$\begin{aligned} &\frac{\hbar}{2mc^2}\left(\frac{\partial}{\partial t} + \frac{e}{\hbar}A_0\right)^2\phi + \frac{\hbar}{2m}\left(\nabla - \mathrm{i}\frac{e}{\hbar}\boldsymbol{A}\right)^2\phi - \frac{mc^2}{2\hbar}\phi + \frac{e}{2m}\boldsymbol{\sigma}\cdot\left(\boldsymbol{B} + \frac{\mathrm{i}}{c}\boldsymbol{E}\right)\phi \\ &\quad = \frac{\hbar}{2mc^2}\left(\frac{\partial}{\partial t} + \frac{e}{\hbar}A_0\right)^2\phi + \frac{\hbar}{2m}\left(\nabla - \mathrm{i}\frac{e}{\hbar}\boldsymbol{A}\right)^2\phi - \frac{mc^2}{2\hbar}\phi + \boldsymbol{\sigma}\cdot\boldsymbol{F}\phi \\ &\quad = \frac{\hbar}{2mc^2}\left(\frac{\partial}{\partial t} + \frac{e}{\hbar}A_0\right)^2\phi + \frac{\hbar}{2m}\left(\nabla - \mathrm{i}\frac{e}{\hbar}\boldsymbol{A}\right)^2\phi - \frac{mc^2}{2\hbar}\phi - \frac{1}{\hbar}\Lambda\phi = 0 \end{aligned} \tag{2.6}$$

where $\boldsymbol{E} = -\nabla A_0$ and $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ are the electric and magnetic fields. Any solution (2.5), (2.6) is a solution of (2.1) and *vice versa* and, therefore, we have the problem of finding the positive energy solutions of (2.6). Fortunately, we are in the Euclidean region where positive frequency solutions vanish exponentially when $t \to +\infty$ while the negative ones explode. Therefore, we add to the boundary condition

$$\phi(t = 0, \boldsymbol{x}) = \phi_0(\boldsymbol{x}) \tag{2.7}$$

a further condition of regularity to the infinity, namely

$$\lim_{t\to+\infty}\phi(t, \boldsymbol{x}) = 0. \tag{2.8}$$

The solution of the Dirichlet problem (2.6), (2.7) and (2.8) is unique in the domain $t > 0$. In some sense, the condition on the behaviour at $t = +\infty$ replaces the more usual condition (in the Minkowski region) which fixes the value of $\partial\phi/\partial t$ at the time $t = 0$ and it has the obvious advantage that (2.8) automatically selects the positive energy solutions. If the matrix $\Lambda(\boldsymbol{x}) = -\hbar\boldsymbol{\sigma}\cdot\boldsymbol{F}(\boldsymbol{x})$, which couples the components of ϕ, were diagonal, one could use a standard probabilistic formula [see 24] to solve the Dirichlet problem (2.6), (2.7), (2.8), nevertheless, in our non-diagonal case, it is possible to generalize such a technique. The result is the following path integral representation (where the contribution of the rest energy has been isolated)

$$\begin{aligned}\phi(t,\boldsymbol{x})\exp(mc^2/\hbar t) &= \exp\{-(t/\hbar)(H_{\mathrm{e}^-} - mc^2)\}\phi_0(\boldsymbol{x})\\ &= \mathbb{E}\left(M(\tau(t))\,\phi_0(\boldsymbol{x}_{\tau(t)})\exp-\frac{S(\tau(t))}{\hbar}\right)\end{aligned} \tag{2.9}$$

with

$$S(\tau) = \mathrm{i}e\int_0^\tau \boldsymbol{A}(\boldsymbol{x}_s)\cdot\mathrm{d}\boldsymbol{x}_s + e\int_0^\tau A_0(\boldsymbol{x}_s)\,\mathrm{d}x_s^0 = e\int_0^\tau \mathcal{A}_\mu(\boldsymbol{x}_s)\,\mathrm{d}x_s^\mu. \tag{2.10}$$

Here $x_s^\mu = (x_s^0, \boldsymbol{x}_s)$ with $\boldsymbol{x}_s = \boldsymbol{x} + \sqrt{(\hbar/m)}\boldsymbol{w}_s$, $x_s^0 = s + \sqrt{(\hbar/mc^2)}w_s^0$ (where $(w_s^0, \boldsymbol{w}_s)$ is a standard four-dimensional Brownian motion) and $\mathcal{A}_\mu = (A^0, \mathrm{i}\boldsymbol{A})$. The random variable $\tau(t)$ is the stochastic time defined by $\tau(t) = \inf\{s \geqslant 0 : x_s^0 = t\} = \inf\{s \geqslant 0 : s + \sqrt{(\hbar/mc^2)}w_s^0 = t\}$ (in other words $\tau(t)$ is the first hitting time of t by x_s^0). Finally, $M(\tau)$ is the (stochastic) 2×2 matrix which solves the first-order differential equation

$$(\mathrm{d}M(s)/\mathrm{d}s) = -(1/\hbar)M(s)\Lambda(\boldsymbol{x}_s) = M(s)\boldsymbol{\sigma}\cdot\boldsymbol{F}(\boldsymbol{x}_s) \tag{2.11}$$

with the initial condition $M_{\alpha\beta}(0) = \delta_{\alpha\beta}$.

In this form, our path integral formula involves only diffusions and it reminds us of the Feynman–Kac–Itô formula for Pauli Hamiltonians [25]. The main difference lies in the fact that an extra Brownian motion w_s^0 appears in (2.9), the deterministic time t is replaced by the Markov time $\tau(t)$ and, moreover, the matrix $M(\tau)$ also depends on the electric field. The derivation of (2.9) is given in appendix A.

One final problem remains to be settled. The solution of the equation (2.11) is the anti-ordered exponential

$$M(\tau) = T^*\exp-\frac{1}{\hbar}\int_0^\tau \Lambda(\boldsymbol{x}_s)\,\mathrm{d}s \tag{2.12}$$

defined as a product of $\exp-(1/\hbar)\Lambda(\boldsymbol{x}_s)\,\mathrm{d}s$ with increasing values of s from the left to the right. Unfortunately, this expression can only be made explicit when $\boldsymbol{F}(\boldsymbol{x})$ has a constant direction. In this case, in fact, the matrices $\Lambda(\boldsymbol{x}_s)$ commute at different times and the anti-ordered exponential becomes an ordinary one. It would also be useful to give a compact representation of (2.12) in the case in which $\boldsymbol{F}(\boldsymbol{x})$ is not constant in direction. This is indeed possible. Using a standard Poisson process $N(s)$, with a probability rate of jump equal to one, we have the following explicit formula for $M(\tau)$

$$M(\tau) = \mathbb{E}\left[\begin{pmatrix}\frac{1}{2}1 + (-1)^{N(\tau)}L_+(\tau) & \frac{1}{2}1 - (-1)^{N(\tau)}L_+(\tau)\\ \frac{1}{2}1 - (-1)^{N(\tau)}L_-(\tau) & \frac{1}{2}1 + (-1)^{N(\tau)}L_-(\tau)\end{pmatrix}\right] \tag{2.13}$$

where the expectation $\mathbb{E}(\cdot)$ is taken with respect to the Poisson process and

$$L_{\pm}(\tau) = \exp\left\{ \int_0^{\tau} [1 \pm (-1)^{N(s)} F_3(\boldsymbol{x}_s)]\, \mathrm{d}s + \int_0^{\tau} \log[F_1(\boldsymbol{x}_s) \pm \mathrm{i}(-1)^{N(s)} F_2(\boldsymbol{x}_s)]\, \mathrm{d}N(s) \right\} \tag{2.14}$$

in which F_1, F_2, F_3 are the three components of the complex vector field $\boldsymbol{F}(\boldsymbol{x})$. The derivation of (2.13), (2.14) is given in appendix B (see also 26). If we put this expression into (2.9), we get the solution of the Dirichlet problem, by means of a path integral representation for the semigroup $\phi(t, \boldsymbol{x}) = \exp\{-(t/\hbar) H_{e^-}\}\phi_0(\boldsymbol{x})$, in the most general case. The contribution of the electromagnetic field is completely explicit since it is entirely contained in the factors $\exp-(S/\hbar)$ and $L_{\pm}$. It turns out that $\exp-(S/\hbar)$ takes into account the interaction of the particle with the field as if it were spinless, while the matrix $M(\tau)$ contains the contributions due to the interaction of the spin with the magnetic and electric fields.

Equations (2.13), (2.14) also hold when $\boldsymbol{F}(\boldsymbol{x})$ has a constant direction but it will be convenient, in this case, to insert directly (2.12) into (2.9). Consider for example an electron in a purely electric field with a constant direction (one can choose $\boldsymbol{E}(\boldsymbol{x}) = (0, 0, E_3(\boldsymbol{x}))$). It turns out, from (2.12), that

$$\phi(t, \boldsymbol{x}) \exp\left\{\frac{mc^2}{\hbar}\right\} t = \mathbb{E}\left(M(\tau(t))\, \phi_0(\boldsymbol{x}_{\tau(t)}) \exp\left\{ -\frac{e}{\hbar} \int_0^{\tau(t)} A_0(\boldsymbol{x}_s)\, \mathrm{d}x_s^0 \right\}\right) \tag{2.15}$$

where $M(\tau(t))$ is the diagonal matrix with upper and lower elements

$$M(\pm 1, \pm 1, \tau(t)) = \exp\left\{ \pm \frac{\mathrm{i}e}{2mc} \int_0^{\tau(t)} E_3(\boldsymbol{x}_s)\, \mathrm{d}s \right\}. \tag{2.16}$$

We observe that the matrix $M(\tau)$ does not reduce to the identity. In fact, in the absence of a magnetic field, $M(\tau)$ still accounts for the spin–orbit interaction. This is at variance with the non-relativistic case.

3. The Hilbert space

Using formula (2.5), which gives χ in terms of ϕ, and the probabilistic formula (2.9), which gives ϕ in terms of ϕ_0, one can reconstruct all of the Dirac spinor ψ. At this point, one could say that the task of giving a path integral solution of the imaginary-time Dirac equation has been accomplished. This is indeed true, but, since we only look for positive energy solutions, we cannot give arbitrary boundary conditions for ψ but we can only give initial conditions which satisfies (2.5). In contrast we have a positive energy solution of (2.6) for any boundary condition $\phi_0(\boldsymbol{x})$. Therefore, it would be useful to redefine the electron Hilbert space so that all the physical quantities can be calculated from the two-component Pauli spinor ϕ only. For positive energy solutions of (2.6), one has $\hbar\partial\phi/\partial t = -H_{e^-}\phi$ and formula (2.5) can be written as

$$\chi = \frac{1}{mc^2}[H_{e^-} + c\boldsymbol{\sigma}\cdot(\mathrm{i}\hbar\nabla + e\boldsymbol{A}) - eA_0]\phi \equiv z\phi \tag{3.1}$$

and ψ as

$$\psi = \begin{pmatrix} 1 \\ z \end{pmatrix} \phi \tag{3.2}$$

where $\binom{1}{z}$ is a 4×2 matrix. The scalar product between two (positive energy) Dirac spinors ψ' and ψ becomes

$$(\psi'|\psi) = \left(\phi' \begin{pmatrix} 1 \\ z^+ \end{pmatrix} \middle| \begin{pmatrix} 1 \\ z \end{pmatrix} \phi \right) = (\phi'|(1 + z^+ z)\phi) \tag{3.3}$$

and the last equality defines the scalar product $\langle\phi'|\phi\rangle$ between the corresponding Pauli spinors ϕ' and ϕ

$$\langle\phi'|\phi\rangle \equiv (\phi'|(1 + z^+ z)\phi) \tag{3.4}$$

which, in turn, defines the electron Hilbert space in our representation. In other words, the eigenvectors of the Hamitonian H_{e^-} are orthonormal with respect the metric $(1 + z^+ z)$ namely, if ψ_m and ψ_n are two (normalized) eigenfunctions of the Dirac Hamiltonian H_D with (positive) energies E_m and E_n respectively, then the corresponding Pauli spinor ϕ_m and ϕ_n satisfy

$$\langle\phi'_m|\phi_n\rangle = (\phi'_m|(1 + z^+ z)\phi_n) = \delta_{mn}. \tag{3.5}$$

It is easy to show that H_{e^-} is Hermitian with respect to the scalar product (3.4), in other words that

$$\langle\phi'|H_{e^-}\phi\rangle = \langle H_{e^-}\phi'|\phi\rangle. \tag{3.6}$$

In fact, by definition

$$H_D\psi = H_D \begin{pmatrix} 1 \\ z \end{pmatrix} \phi = \begin{pmatrix} 1 \\ z \end{pmatrix} H_{e^-}\phi \tag{3.7}$$

and therefore

$$\begin{aligned} \langle\phi'|H_{e^-}\phi\rangle &= (\phi'|(1 + z^+ z) H_{e^-}\phi) = \left(\phi' \begin{pmatrix} 1 \\ z^+ \end{pmatrix} \middle| \begin{pmatrix} 1 \\ z \end{pmatrix} H_{e^-}\phi \right) \\ &= \left(\phi' \begin{pmatrix} 1 \\ z^+ \end{pmatrix} \middle| H_D \begin{pmatrix} 1 \\ z \end{pmatrix} \phi \right) = \left(\phi' \begin{pmatrix} 1 \\ z^+ \end{pmatrix} H_D \middle| \begin{pmatrix} 1 \\ z \end{pmatrix} \phi \right) = \langle H_{e^-}\phi'|\phi\rangle. \end{aligned} \tag{3.8}$$

It is also possible to give the rule of tranformation for all operators O, corresponding to electron physical observables, from the Dirac representation to ours. One has

$$(\psi'|O\psi) = \left(\phi' \begin{pmatrix} 1 \\ z^+ \end{pmatrix} \middle| O \begin{pmatrix} 1 \\ z \end{pmatrix} (1 + z^+ z)^{-1}(1 + z^+ z)\phi \right). \tag{3.9}$$

Therefore, in our representation, the operator O becomes

$$O' = \begin{pmatrix} 1 \\ z^+ \end{pmatrix} O \begin{pmatrix} 1 \\ z \end{pmatrix} (1 + z^+ z)^{-1}. \tag{3.10}$$

One would also know what is the link between our representation and the Foldy-Wothuysen one. In both representations there is a positive Hamiltonian for the electron, but, in the Foldy–Wouthuysen case, the scalar product defining the Hilbert space is the ordinary one. If we define $\tilde{\phi} = \sqrt{1+z^+z}\phi$, we immediately see that

$$\langle\phi'|\phi\rangle = (\phi'|(1+z^+z)\phi) = (\tilde{\phi}'|\tilde{\phi}). \tag{3.11}$$

This transformation gives the required link between our representation and the Foldy–Wouthuysen one. In particular, the electron Hamiltonian transforms as

$$\tilde{H}_{e^-} = (1+z^+z)^{1/2} H_{e^-} (1+z^+z)^{-1/2} \tag{3.12}$$

where $\tilde{H}_{e^-}$ is the Foldy–Wouthuysen electron Hamiltonian (which is Hermitian with respect to the ordinary scalar product). The two operators $\tilde{H}_{e^-}$ and H_{e^-} have the same spectrum and, furthermore, coincide when the external field is purely magnetic. In this case, in fact, z and H_{e^-} commute and one has (see [19, 20])

$$H_{e^-} = \sqrt{c^2(\sigma\cdot(i\hbar\nabla + eA))^2 + m^2c^4}. \tag{3.13}$$

The Hamiltonian semigroup for the positron can be obtained exactly in the same way by looking at the Dirichlet problem (2.6) in the domain $t < 0$. The result is a path integral representation which is identical to (2.9) except that the four-potential A^μ is replaced by $-A^\mu$. A transformation analogous to (3.12) then gives the link with the positron Hamiltonian which emerges from the Foldy–Wouthuysen transformation.

4. Discussion

Formula (2.9) solves equation (2.6). The Klein–Gordon equation is very similar to (2.6) with only two differences: the matrix Λ is absent and the function ϕ is a scalar. Therefore, the same argument that applies to positive energy solutions of Dirac equation (in form (2.6)) also applies to the positive frequency solutions of Klein–Gordon equation. One has, in fact, for the latter

$$\phi(t,x)\exp\left\{\frac{mc^2}{\hbar}\right\}t = \mathbb{E}\left(\phi_0(x_{\tau(t)})\exp\left\{-\frac{S(\tau(t))}{\hbar}\right\}\right) \tag{4.1}$$

where $M(\tau(t))$ has disappeared. This result has already been reported in a previous paper [27], where a detailed discussion of the associated Hilbert space was also given. Some words about the non-relativistic limit of formulae (2.9) and (4.1) are in order at this point. We first remark that when $c \to +\infty$ the stochastic process $x_s^0 = s + \sqrt{(\hbar/mc^2)}w_s^0$ converges to the deterministic value s and, therefore, the stochastic time $\tau(t)$ and the deterministic time t coincide in this limit. This fact was stated more precisely [28]. By subtracting the rest energy, formulae (2.9) and formula (4.1) become, respectively, in the non-relativistic limit

$$\phi(t,x) = \mathbb{E}\left(M(t)\phi_0(x_t)\exp\left\{-\frac{S(t)}{\hbar}\right\}\right) \tag{4.2}$$

and

$$\phi(t,\boldsymbol{x}) = \mathbb{E}\left(\phi_0(\boldsymbol{x}_t)\exp\left\{-\frac{S(t)}{\hbar}\right\}\right) \tag{4.3}$$

$S(t)$ is now

$$S(t) = \mathrm{i}e\int_0^t \boldsymbol{A}(\boldsymbol{x}_s)\cdot \mathrm{d}\boldsymbol{x}_s + e\int_0^t A_0(\boldsymbol{x}_s)\,\mathrm{d}s \tag{4.4}$$

because the integration with respect to $\mathrm{d}x_s^0$ has been replaced by an integration with respect to s. The matrix M is still given by (2.12) but now only contains the magnetic field because $\boldsymbol{F} = (e/2m)(\boldsymbol{B} + (\mathrm{i}/c)\boldsymbol{E})$ reduces to $(e/2m)\boldsymbol{B}$ in the non-relativistic limit. Formula (4.2) is precisely the Feynman–Kac–Itô formula for a non-relativistic electron described by a Pauli Hamiltonian (see [25, 26]) and (4.3) is the Feynman–Kac–Itô formula for a Schrödinger particle. When the magnetic field is absent both formulae reduce to

$$\phi(t,\boldsymbol{x}) = \mathbb{E}\left(\phi_0(\boldsymbol{x}_t)\exp\left\{-\frac{e}{\hbar}\int_0^t A_0(\boldsymbol{x}_s)\,\mathrm{d}s\right\}\right) \tag{4.5}$$

which is the ordinary Feynman–Kac formula.

Appendix A

Formula (2.9) is obtained in this appendix by improving standard methods for the probabilistic solution of Dirichlet problems [24].

We start by defining the process $s \mapsto u(s)$ through $u(s) = u(t - x_s^0, \boldsymbol{x}_s)$ where $u(t,\boldsymbol{x})$ is a given two-component smooth function, regular for $t \to +\infty$, and the spacetime diffusion $s \mapsto (x_s^0, \boldsymbol{x}_s)$ is the same as the one which appears in all previous formulae. From Itô's lemma, one has (here $\hbar = m = c = 1$)

$$\mathrm{d}u = \nabla u \cdot \mathrm{d}\boldsymbol{w}_s - \frac{\partial u}{\partial t}\,\mathrm{d}w_s^0 - \frac{\partial u}{\partial t}\,\mathrm{d}s + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\,\mathrm{d}s + \frac{1}{2}\Delta u\,\mathrm{d}s. \tag{A.1}$$

In the same way, $C(s) = \exp -S(s)$, where $S(s)$ is defined by (2.10), satisfies

$$\begin{aligned}\mathrm{d}C(s) = -C(s)(\mathrm{i}e\boldsymbol{A}(\boldsymbol{x}_s)\cdot \mathrm{d}\boldsymbol{w}_s + eA_0(\boldsymbol{x}_s)\,\mathrm{d}w_s^0 + \\ + eA_0(\boldsymbol{x}_s)\,\mathrm{d}s + \tfrac{1}{2}e^2|\boldsymbol{A}(\boldsymbol{x}_s)|^2\,\mathrm{d}s - \tfrac{1}{2}e^2A_0^2(\boldsymbol{x}_s)\,\mathrm{d}s)\end{aligned} \tag{A.2}$$

where we have explicitly made use of our choice of gauge. By defining $D(s)$ as $D(s) = \exp -x_s^0$, one also has

$$\mathrm{d}D(s) = -D(s)(\mathrm{d}w_s^0 + \tfrac{1}{2}\mathrm{d}s) \tag{A.3}$$

and finally $M(s)$, defined by (2.12), is differentiable in the ordinary sense and, therefore

$$\mathrm{d}M(s) = -M(s)\Lambda(\boldsymbol{x}_s)\,\mathrm{d}s. \tag{A.4}$$

From these four stochastic differentials one obtains

$$\begin{aligned}
&\mathrm{d}(M(s)C(s)D(s)u(s)) \\
&\quad = M(s)C(s)D(s)\left[\frac{1}{2}\left(\frac{\partial}{\partial t}+eA_0\right)^2 u+\frac{1}{2}(\nabla-\mathrm{i}e\boldsymbol{A})^2 u-\frac{1}{2}u-\Lambda u\right]\mathrm{d}s \\
&\qquad + M(s)C(s)D(s)\left[-\left(\frac{\partial u}{\partial t}+eA_0 u-u\right)\mathrm{d}w_s^0+(\nabla u-\mathrm{i}e\boldsymbol{A}u)\,\mathrm{d}\boldsymbol{w}_s\right].
\end{aligned} \tag{A.5}$$

The integration of this identity between 0 and $\tau(t)=\inf\{s \geqslant 0 : x_s^0 = t\}$ provides the formula

$$\begin{aligned}
&M(\tau(t))C(\tau(t))u(0,\boldsymbol{x}_{\tau(t)})\exp\{-t\} \\
&\quad = u(t,\boldsymbol{x})+\int_0^{\tau(t)} M(s)C(s)D(s)\left[\frac{1}{2}\left(\frac{\partial}{\partial t}+eA_0\right)^2 u\right. \\
&\qquad \left.+\frac{1}{2}(\nabla-\mathrm{i}e\boldsymbol{A})^2 u-\frac{1}{2}u-\Lambda u\right]\mathrm{d}s \\
&\qquad +\int_0^{\tau(t)} M(s)C(s)D(s) \\
&\qquad \times\left[-\left(\frac{\partial u}{\partial t}+eA_0 u-u\right)\mathrm{d}w_s^0+(\nabla u-\mathrm{i}e\boldsymbol{A}u)\cdot\mathrm{d}\boldsymbol{w}_s\right]
\end{aligned} \tag{A.6}$$

where we have exploited the equalities $u(\tau(t)) = u(t-x^0_{\tau(t)}, \boldsymbol{x}_{\tau(t)}) = u(0,\boldsymbol{x}_{\tau(t)})$ and $D(\tau(t)) = \exp\{-t\}$ because, $x^0_{\tau(t)} = t$ by definition. We now take the expectation of this equality. If the upper integration limit were a number, the expectation containing Itô's integral would vanish. This property of Itô integrals still holds if the upper integration limit is a Markov time with finite expectation, a class which includes $\tau(t)$. We have, therefore,

$$\begin{aligned}
&\mathbb{E}(M(\tau(t))u(0,\boldsymbol{x}_{\tau(t)})C(\tau(t)))\exp\{-t\} \\
&\quad = u(t,\boldsymbol{x})+\mathbb{E}\left(\int_0^{\tau(t)} M(s)C(s)D(s)\right. \\
&\qquad \left.\times\left[\frac{1}{2}\left(\frac{\partial}{\partial t}+eA_0\right)^2 u+\frac{1}{2}(\nabla-\mathrm{i}e\boldsymbol{A})^2 u-\frac{1}{2}u-\Lambda u\right]\mathrm{d}s\right).
\end{aligned} \tag{A.7}$$

Now, if $u(t,\boldsymbol{x})$ satisfies the equation:

$$\frac{1}{2}\left(\frac{\partial}{\partial t}+eA_0\right)^2 u+\frac{1}{2}(\nabla-\mathrm{i}e\boldsymbol{A})^2 u-\frac{1}{2}u-\Lambda u=0 \tag{A.8}$$

for $t > 0$, then the equality (A.7) reduces to

$$u(t,\boldsymbol{x})\exp t=\mathbb{E}(M(\tau(t))u(0,\boldsymbol{x}_{\tau(t)})C(\tau(t))) \tag{A.9}$$

which coincides with (2.9) when the dimensional constants $\hbar$, c and m are explicitly inserted.

Appendix B

In this appendix we show that the stochastic matrix $M(\tau)$, given by the probabilistic formula (2.13), satisfies the differential equation (2.11). We first rewrite all expressions in a scalar form. After having defined

$$M(\tau) = \begin{pmatrix} M(1,1,\tau) & M(1,-1,\tau) \\ M(-1,1,\tau) & M(-1,-1,\tau) \end{pmatrix} \tag{B.1}$$

(2.13) becomes

$$M(a,b,\tau) = \mathbb{E}\bigg(\frac{1+ab(-1)^{N(\tau)}}{2} \exp\left\{\int_0^\tau [1+a(-1)^{N(s)}F_3(\boldsymbol{x}_s)]\,\mathrm{d}s\right\} \times \exp\left\{\int_0^\tau \log[F_1(\boldsymbol{x}_s)+\mathrm{i}a(-1)^{N(s)}F_2(\boldsymbol{x}_s)]\,\mathrm{d}N(s)\right\}\bigg) \tag{B.2}$$

where a and b take the values ± 1 and, therefore, $M(a,b,\tau)$ represents the four entries of $M(\tau)$. In order to find the differential equation satisfied by $M(\tau)$ we rewrite this expression for a larger time $\tau+\Delta\tau$. We have

$$\begin{aligned} M(a,b,\tau+\Delta\tau) = \mathbb{E}\bigg[&\frac{1+ab(-1)^{N(\tau)}(-1)^{N'(\Delta\tau)}}{2} \\ &\times \exp\left\{\int_0^\tau [1+a(-1)^{N(s)}F_3(\boldsymbol{x}_s)]\,\mathrm{d}s\right\} \\ &\times \exp\left\{\int_0^\tau \log[F_1(\boldsymbol{x}_s)+\mathrm{i}a(-1)^{N(s)}F_2(\boldsymbol{x}_s)]\,\mathrm{d}N(s)\right\} \\ &\times \exp\left\{\int_0^{\Delta\tau} [1+a(-1)^{N(\tau)}(-1)^{N'(s)}F_3(\boldsymbol{x}_{\tau+s})]\,\mathrm{d}s\right\} \\ &\times \exp\bigg\{\int_0^{\Delta\tau} \log[F_1(\boldsymbol{x}_{\tau+s}) \\ &+\mathrm{i}a(-1)^{N(\tau)}(-1)^{N'(s)}F_2(\boldsymbol{x}_{\tau+s})]\,\mathrm{d}N'(s)\bigg\}\bigg] \end{aligned} \tag{B.3}$$

where, for $\tau \leqslant s \leqslant \tau+\Delta\tau$ $N(s) = N(\tau)+N(s)-N(\tau) \equiv N(\tau)+N'(s-\tau)$. The processes N and N' are independent and one can take the expectation with respect to N'. In order to calculate this expectation, it is sufficient to use three facts: first, in the infinitesimal time interval $\Delta\tau$, the process N' makes a jump with probability $\Delta\tau$ and no jumps with probability $1-\Delta\tau$; second, the expectation contains the factor $[1+ab(-1)^{N(\tau)}(-1)^{N'(\Delta\tau)}]$ which enforces the value b for $a(-1)^{N(\tau)}$ when no jumps occur and $-b$ in the case of a jump; and third, the stochastic integral

$$\int_0^{\Delta\tau} \log[F_1(\boldsymbol{x}_{\tau+s})+\mathrm{i}a(-1)^{N(\tau)}(-1)^{N'(s)}F_2(\boldsymbol{x}_{\tau+s})]\,\mathrm{d}N'(s) \tag{B.4}$$

is zero when there is no jump and takes the value

$$\log[F_1(\boldsymbol{x}_\tau)+\mathrm{i}a(-1)^{N(\tau)}F_2(\boldsymbol{x}_\tau)] \tag{B.5}$$

when there is a jump (the Poisson process $s \mapsto N(s)$ is assumed to be left-continous). Therefore, formula (B.3) becomes

$$M(a,b,\tau+\Delta\tau) = (1+bF_3\Delta\tau)M(a,b,\tau) + (F_1 - \mathrm{i}bF_2)M(a,b,\tau)\Delta\tau \qquad \text{(B.6)}$$

and, by remembering that the matrix Λ is given by

$$\Lambda = -\hbar \begin{pmatrix} F_3 & F_1 - \mathrm{i}F_2 \\ F_1 + \mathrm{i}F_2 & -F_3 \end{pmatrix} \qquad \text{(B.7)}$$

one can see that equations (B.6) and (2.11) coincide.

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